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Composite models in general relativity

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Abstract. The stability of static spherically symmetrical fluid spheres, consisting of a core and an envelope, is investigated on the basis of general relativity. It is assumed that the core consists of ideal gas and radiation, in which the ratio β of the gas pressure to the total pressure is a small constant, and that the envelope consists of an adiabatic gas. Numerical analysis indicates that for a given $\sigma = p_c/\rho_{g_c}c^2$ (the ratio of the central pressure to the central rest-energy density) the stability of such a sphere depends strongly on the position of the interface separating the core from the envelope, the sphere being stable for a greater range of values of σ the closer the interface is to the centre.

1. Introduction

Iben (1963), in studying the stability of a succession of static fluid spheres, drew attention to the importance of the binding energy in determining the behaviour of a given model. Since then problems of stability of quasi-static configurations have been analysed from this point of view. Tooper (1964), when considering static general relativistic polytropic fluid spheres, found that, although a negative binding energy is a *necessary* condition for instability it is not a *sufficient* condition. Subsequently Tooper (1965a) has shown that for small values of σ at least the binding energy is a maximum at the point of onset of instability; for larger values of σ the configuration is unstable against radial perturbations and for a polytropic sphere of index $n = 3$ the sphere is unstable for *all* values of σ, the binding energy being always negative.

Without going into details we may state that even typical stars on the main sequence in the Hertzprung–Russel diagram contain certain inhomogeneities. The core—where the thermonuclear transmutation of hydrogen occurs—is represented by one set of equations whereas the envelope is characterized by another set. White dwarfs and red giants exhibit a composite nature to an even greater degree. A white dwarf is expected to consist of relativistic degenerate matter near the centre but the degeneracy in the outer parts, at least, is expected to be non-relativistic.

It is shown in the present paper that the introduction of an ideal gas envelope has the effect of increasing the binding energy and raising the value of σ at which this binding energy is a maximum thus rendering the sphere more stable against radial perturbations.

2. Basic equations

2.1. Field equations

Using a co-moving coordinate system at rest with respect to the fluid the components of the energy–momentum tensor may be written

$$T_1^1 = T_2^2 = T_3^3 = -p \qquad T_4^4 = \rho c^2 \tag{2.1}$$

and with the metric in the form

$$ds^2 = -e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + e^\nu dt^2 \tag{2.2}$$

where $\lambda = \lambda(r)$, $\nu = \nu(r)$ are functions of r only, the field equations may be written

$$\frac{8\pi G p}{c^4} = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} \tag{2.3}$$

$$\frac{8\pi G \rho}{c^2} = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} \tag{2.4}$$

and

$$\frac{dp}{dr} = -(p + \rho c^2)\frac{\nu'}{2}.$$

Equation (2.4) may be integrated immediately by writing it in the form

$$\frac{8\pi G \rho r^2}{c^2} = 1 - (r\, e^{-\lambda})' \tag{2.5}$$

and consequently

$$e^{-\lambda} = 1 - \frac{2G}{rc^2}\int_0^r 4\pi\rho r^2\, dr. \tag{2.6}$$

Defining the mass inside the radius r (arising from all causes) as measured by an external observer to be M_r so that

$$M_r = \int_0^r 4\pi\rho r^2\, dr \tag{2.7}$$

equation (2.6) may be written as

$$e^{-\lambda} = 1 - \frac{2GM_r}{rc^2}. \tag{2.8}$$

Equations (2.3), (2.4) and (2.5) are three independent equations in four unknowns λ, ν, p and ρ, and thus in order to solve them completely it is necessary to introduce a further condition. This usually takes the form of an equation of state $p = p(\rho)$ connecting the pressure p with the density ρ. However, other approaches have been used (Einstein 1939, Tolman 1939, Thompson and Whitrow 1967), which either supplement or replace the equation of state but these will not be considered in this paper.

2.2. *Equation of state*

A relativistic equation of state proposed by Tooper (1965a) for a perfect gas undergoing an adiabatic process is given by

$$p = K\rho_g^{1+1/n} \qquad \rho c^2 = \rho_g c^2 + np \tag{2.9}$$

where ρ_g is the density of the rest-mass of the gas and $n = 1/(\gamma - 1)$. The velocity of sound for this equation of state, unlike that for a polytrope, is always less than light provided that the index $n \geqslant 1$. (This equation of state will be used in the envelope.)

Another equation of state (which will be used in the core) is that of a mixture of perfect gas and isotropic radiation at a temperature T (Tooper 1965b).

The total pressure may be expressed as

$$p = p_g + p_r$$

where

$$p_g = \left(\frac{k}{\mu H}\right) T \rho_g \qquad \text{and} \qquad p_r = \tfrac{1}{3} a T^4 \tag{2.10}$$

are the pressures of the gas and radiation respectively. Here ρ_g is the gas density, k is Boltzmann's constant, μ is the molecular weight and H is the mass of a proton.

Since ρ is the total density, that is the sum of the densities of the rest mass of the gas, the energy content of the microscopic kinetic energy of the gas and the energy of the radiation, it follows that

$$\rho c^2 = \rho_g c^2 + \frac{p_g}{\gamma - 1} + 3p_r$$

where γ is the ratio of the specific heats of the gas. Now if we define β as the ratio of the gas pressure to the total pressure we have

$$p_g = \beta p \qquad \text{and} \qquad p_r = (1-\beta)p$$

and consequently

$$\beta p = \left(\frac{k}{\mu H}\right) \rho_g T \qquad (1-\beta)p = \frac{1}{3} a T^4. \tag{2.11}$$

If β is constant, elimination of T between the above equations gives

$$p = K(\beta)\rho_g^{4/3} \qquad \text{where} \qquad K(\beta) = \left\{\left(\frac{k}{\mu H}\right)^4 \frac{3}{a} \frac{1-\beta}{\beta^4}\right\}^{1/3}$$

and so the equation of state in parametric form becomes

$$p = K(\beta)\rho_g^{4/3} \qquad \rho c^2 = \rho_g c^2 + \frac{\beta}{1-\gamma} K(\beta)\rho_g^{4/3} + 3(1-\beta)K(\beta)\rho_g^{4/3} \tag{2.12}$$

giving the total energy-density ρc^2 in terms of the pressure p.

2.3. The composite model

In order to avoid unnecessary repetition, the equation of state will be taken in the general form

$$p = K\rho_g^{1+1/n} \qquad \rho c^2 = \rho_g c^2 + Ap \tag{2.13}$$

where the appropriate values of the constants A and n will be chosen to correspond to the particular equation of state under consideration.

2.3.1. Core. The equation of state for the core is given by equation (2.12) or equivalently by (2.13) with $n = 3$ and $A = \{\beta/(\gamma - 1)\} + 3(1-\beta)$.

2.3.2. Envelope. The equation of state in the envelope is taken to be equation (2.9) or equation (2.13) with $A = n, (n < 3)$.

The distance from the centre of the configuration at which (2.12) must be replaced by (2.9) we will define as the interfacial radius. Clearly the envelope as such would not exist if we were to assume that β is the same constant throughout the configuration, for in this case the index n in equation (2.13) would be equal to 3 and A would be given by $A = \{\beta/(\gamma-1)\}+3(1-\beta)$. However, except possibly in the case of extremely massive stars ($\geqslant 10^8$ solar masses), β is unlikely to be constant throughout the model. In fact Hoyle and Fowler (1964) have shown that for polytropes in which β is small it will depend on the polytropic variable θ according to the relation

$$\beta \sim \frac{1}{\mu}\left\{\frac{3}{4\pi}(n+1)^3\left(\frac{k}{H}\right)^4\frac{1}{aG^3}\right\}^{1/4}\left(\frac{V(\xi_s)}{M}\right)^{1/2}\theta^{(n-3)/4} \tag{2.14}$$

where the symbols have their customary meanings. It follows that if β is small then only for a polytrope of index 3 is β a constant throughout the model, being in fact given by

$$\beta \sim \frac{4{\cdot}3}{\mu}\left(\frac{M_s}{M}\right)^{1/2} \tag{2.15}$$

where M_s is one solar mass.

An alternative way to obtain equation (2.14) is to use two expressions given by Milne (1930) for β and for the total mass M giving

$$\frac{\beta^4}{1-\beta} = \frac{3(n+1)^3}{4\pi}\left(\frac{k}{\mu H}\right)^4\frac{1}{aG^3}\left(\frac{V(\xi_s)}{M}\right)^2\theta^{n-3}. \tag{2.16}$$

For $n < 3$ it follows that near the surface (where $\theta \to 0$) the right-hand side of (2.16) is very large, which of course means that β is close to unity, and hence the radiation pressure becomes small compared with the gas pressure. The equation of state may then be taken as that for an adiabatic sphere (2.9), since $\beta = 1$ at the surface.

2.4. Characteristic equation for core and envelope

2.4.1. Core. We define σ by

$$\sigma = \frac{p_c}{\rho_{gc}c^2} = \frac{K(\beta)}{c^2}\rho_{gc}^{1/3} \tag{2.17}$$

then in terms of the dimensionless variables ξ, θ and $V(\xi)$ defined by

$$\rho_g = \rho_{gc}\theta^3 \tag{2.18}$$

$$r = \alpha\xi$$

$$M_r = 4\pi\rho_{gc}\alpha^3 V(\xi) \tag{2.19}$$

where r is the radius and M_r is the mass inside the radius r and where $\alpha^2 = \sigma c^2/\pi G\rho_{gc}$ the equation of hydrostatic equilibrium becomes

$$\frac{1-8\sigma V(\xi)/\xi}{1+(1+A)\sigma\theta}\left(\xi^2\frac{d\theta}{d\xi}\right)+V(\xi)+\sigma\xi^3\theta^4 = 0 \tag{2.20}$$

and

$$\frac{dV}{d\xi} = \xi^2\theta^3(1+A\sigma\theta) \tag{2.21}$$

which are to be solved subject to the usual boundary conditions

$$\theta(0) = 1 \qquad V(0) = 0 \qquad \left(\frac{d\theta}{d\xi} \to 0 \quad \text{as} \quad \xi \to 0\right). \tag{2.22}$$

The solutions are relativistic generalizations of the usual Lane–Emden solutions but, unlike the case of complete models, the surface, the total mass, the radius, etc can only be defined when the equation of state (2.12) holds throughout. We can, however, define the interfacial values (denoted by subscript i) of these quantities.

At the interface (where the envelope joins onto the core) the radius r_i is given by

$$r_i = \alpha\xi_i \tag{2.23}$$

and the pressure and density of the rest-mass of the gas by

$$\rho_{g_i} = \rho_{g_c}\theta_i^3$$

and

$$p_i = p_c\theta_i^4 = K(\beta)\rho_{g_c}^{4/3}\theta_i^4 \tag{2.24}$$

hence the total energy density at the interface is given by

$$\rho_i c^2 = \rho_{g_i}c^2 + Ap_i = \rho_{g_c}\theta_i^3c^2 + \frac{\beta}{\gamma-1}K(\beta)\rho_{g_c}^{4/3}\theta_i^4 + 3(1-\beta)K(\beta)\rho_{g_c}^{4/3}\theta_i^4 \tag{2.25}$$

and the mass inside this interface is

$$M_i = 4\pi\rho_{g_c}\alpha^3V(\xi_i). \tag{2.26}$$

2.4.2. Envelope. For the envelope we shall take the equation of state to be (2.13) with a general $n = n_1 \leqslant 3$ and with A replaced by A_1. To avoid confusion with the corresponding quantities in the core the variables ξ, θ and $V(\xi)$ in the envelope will be replaced by η, ϕ, $V_1(\eta)$ respectively, and the envelope values of the parameters σ, n, α will be indicated by the subscript 1.

By analogy with the analysis in the core it is convenient to introduce a new variable ϕ defined by

$$\rho_g = \rho_{g_c}\phi^{n_1} \tag{2.27}$$

where the value ρ_{g_c} is identical with that in equation (2.18). We also define

$$\sigma_1 = \frac{K_1\rho_{g_c}^{1/n_1}}{c^2} \tag{2.28}$$

and write

$$p = K_1\rho_{g_c}^{1+1/n_1}\phi^{n_1+1} \tag{2.29}$$

and

$$r = \alpha_1\eta. \tag{2.30}$$

For the mass inside radius r we have

$$M_r = 4\pi\rho_{g_c}\alpha_1^3 V_1(\eta) \tag{2.31}$$

where

$$\alpha_1^2 = \frac{(n_1+1)\sigma_1 c^2}{4\pi G\rho_{g_c}}. \tag{2.32}$$

The equations of hydrostatic equilibrium for the envelope become

$$\frac{1-2(n_1+1)\sigma_1 V_1(\eta)/\eta}{1+(1+A_1)\sigma_1\phi}\eta^2\frac{d\phi}{d\eta}+V_1(\eta)+\sigma_1\eta^3\phi^{n_1+1} = 0 \tag{2.33}$$

and

$$\frac{dV_1}{d\eta} = \eta^2\phi^{n_1}(1+A_1\sigma_1\phi). \tag{2.34}$$

Although in general the required solutions of the differential equations (2.33) and (2.34) will not, in this case, be the usual generalizations of the Lane–Emden solutions since they do not extend to the centre, we can nevertheless readily define the total mass, the radius, etc of the model. The outer surface is taken to be that radius $r = R$ where the pressure vanishes. In other words, the surface corresponds to the smallest positive value η_s for which

$$\phi(\eta_s) = 0 \tag{2.35}$$

and its radius is given by

$$R = \alpha_1\eta_s. \tag{2.36}$$

Similarly the total mass will be given by

$$M = 4\pi\rho_{g_c}\alpha_1^3 V_1(\eta_s). \tag{2.37}$$

At the interface the value r_i of the radius will be

$$r_i = \alpha_1\eta_i \tag{2.38}$$

and the interfacial values of the pressure, rest-mass density of the gas and the total energy density will be, respectively

$$p_i = K_1\rho_{g_c}^{1+1/n_1}\phi_i^{n_1+1} \tag{2.39}$$

$$\rho_{g_{i1}} = \rho_{g_c}\phi_i^{n_1} \tag{2.40}$$

and

$$\rho_i c^2 = \rho_{g_{i_1}}c^2 + A_1 p_i. \tag{2.41}$$

We may also express the mass inside the interfacial radius r_i by

$$M_i = 4\pi\rho_{g_c}\alpha_1^3 V_1(\eta_i). \tag{2.42}$$

2.5. Interfacial boundary conditions

Since the pressure is to be continuous everywhere and in particular at the interface then the values of this quantity given by equations (2.12) and (2.9) must be identical.

Table 1. Summary of solutions of equations (2.33) and (2.34) for various ξ_i

$n_1 = 1$

ξ_i	θ_i	$V(\xi_i)$	η_i	ϕ_i	$V_1(\eta_i)$	η	ϕ	$V_1(\eta)$
0·0	1·0	0·0	0·0	1·0	0·0	1·0	0·84	0·30
0·0						2·0	0·45	1·74
0·0						3·0	0·05	3·11
0·0						3·14	0·0	3·14
0·5	0·96	0·04	0·68	0·88	0·09	0·9	0·84	0·2
0·5						1·5	0·64	0·82
0·5						2·0	0·44	1·6
0·5						3·1	0·02	3·0
1·0	0·85	0·25	1·21	0·62	0·44	1·3	0·59	0·54
1·0						1·5	0·52	0·76
1·0						2·0	0·35	1·4
1·0						3·0	0·03	2·5
1·5	0·72	0·63	1·53	0·37	0·66	1·6	0·34	0·75
1·5						2·0	0·23	1·1
1·5						2·5	0·10	1·5
1·5						2·9	0·01	1·7
2·0	0·58	1·05	1·65	0·20	0·58	1·74	0·17	0·6
2·0						2·0	0·12	0·8
2·0						2·4	0·04	0·9
2·0						2·7	0·003	1·0
2·5	0·46	1·40	1·63	0·10	0·39	1·7	0·08	0·4
2·5						2·0	0·04	0·48
2·5						2·2	0·02	0·51
2·5						2·4	0·004	0·53
3·0	0·36	1·66	1·52	0·05	0·22	1·6	0·04	0·22
3·0						1·7	0·03	0·23
3·0						1·8	0·02	0·24
3·0						2·1	0·002	0·25

$n_1 = 1{\cdot}5$

ξ_i	θ_i	$V(\xi_i)$	η_i	ϕ_i	$V_1(\eta_i)$	η	ϕ	$V_1(\eta)$
0·0	1·0	0·0	0·0	1·0	0·0	1·0	0·84	0·29
0·0						2·0	0·49	1·49
0·0						3·0	0·16	2·56
0·0						3·65	0·0	2·71
0·5	0·96	0·04	0·6	0·92	0·07	1·0	0·80	0·28
0·5						2·0	0·48	1·40
0·5						3·0	0·15	2·50
0·5						3·6	0·01	2·67
1·0	0·85	0·25	1·06	0·73	0·3	1·2	0·69	0·40
1·0						2·0	0·45	1·20
1·0						3·0	0·17	2·2
1·0						3·7	0·02	2·4
1·5	0·72	0·63	1·31	0·52	0·42	1·5	0·46	0·55
1·5						2·0	0·34	0·96
1·5						3·0	0·13	1·64
1·5						3·9	0·001	1·8

Table 1. (cont)

$n_1 = 1{\cdot}5$ (cont.)								
ξ_i	θ_i	$V(\xi_i)$	η_i	ϕ_i	$V_1(\eta_i)$	η	ϕ	$V_1(\eta)$
2·0	0·58	1·05	1·39	0·34	0·34	1·6	0·3	0·42
2·0						2·0	0·22	0·65
2·0						3·0	0·11	1·0
2·0						4·1	0·006	1·2
2·5	0·46	1·40	1·32	0·21	0·2	1·5	0·2	0·24
2·5						2·0	0·15	0·33
2·5						3·0	0·08	0·53
2·5						4·4	0·005	0·68
3·0	0·36	1·66	1·2	0·13	0·10	2·0	0·08	0·17
3·0						3·0	0·05	0·27
3·0						4·0	0·02	0·35
3·0						5·2	0·001	0·38

$n_1 = 2$								
ξ_i	θ_i	$V(\xi_i)$	η_i	ϕ_i	$V_1(\eta_i)$	η	ϕ	$V_1(\eta)$
0·0	1·0	0·0	0·0	1·0	0·0	1·0	0·85	0·27
0·0						2·0	0·52	1·30
0·0						3·0	0·24	2·16
0·0						4·35	0·0	2·41
0·5	0·96	0·04	0·5	0·94	0·04	1·0	0·8	0·3
0·5						2·0	0·5	1·2
0·5						3·0	0·25	2·1
0·5						4·3	0·003	2·4
1·0	0·85	0·25	0·99	0·8	0·24	2·0	0·5	1·2
1·0						3·0	0·23	2·0
1·0						4·0	0·06	2·2
1·0						4·48	0·006	2·27
1·5	0·72	0·63	1·25	0·61	0·36	2·0	0·42	0·9
1·5						3·0	0·23	1·5
1·5						4·0	0·09	1·86
1·5						4·9	0·001	1·9
2·0	0·58	1·05	1·34	0·44	0·32	2·0	0·3	0·6
2·0						3·0	0·2	1·0
2·0						4·0	0·1	1·3
2·0						5·7	0·002	1·44
2·5	0·46	1·40	1·30	0·31	0·21	3·0	0·17	0·60
2·5						4·0	0·1	0·80
2·5						5·0	0·06	0·98
2·5						7·1	0·001	1·05
3·0	0·36	1·66	1·2	0·21	0·12	3·0	0·12	0·3
3·0						5·0	0·06	0·6
3·0						7·0	0·02	0·78
3·0						9·04	0·003	0·81

Thus

$$p_i = K(\beta)\rho_{g_i}^{4/3} = K_1\rho_{g_i}^{1+1/n_1}. \qquad (2.43)$$

Also from the definitions of σ and σ_1 we have

$$\frac{\sigma_1}{\sigma} = \frac{K_1\rho_{g_c}^{1/n_1}}{K(\beta)\rho_{g_c}^{1/3}} = \frac{K_1}{K(\beta)}(\rho_{g_c})^{1/n_1 - 1/3}. \qquad (2.44)$$

Hence from equations (2.43) and (2.44)

$$\frac{\sigma_1}{\sigma} = \frac{\rho_{g_i}^{4/3}}{\rho_{g_{i_1}}^{1+1/n_1}}\rho_{g_c}^{1/n_1 - 1/3} \qquad (2.45)$$

which becomes, on using the definitions of θ and ϕ

$$\frac{\sigma_1}{\sigma} = \frac{\theta_i^4}{\phi_i^{n_1+1}}. \qquad (2.46)$$

Since at the interface the respective values of r and M_r given by equations (2.23), (2.38) and (2.26), (2.42) must be identical, it follows that

$$r_i = \alpha\xi_i = \alpha_1\eta_i \qquad (2.47)$$

and

$$M_i = 4\pi\rho_{g_c}\alpha^3 V(\xi_i) = 4\pi\rho_{g_c}\alpha_1^3 V_1(\eta_i) \qquad (2.48)$$

and hence

$$\alpha^3 V(\xi_i) = \alpha_1^3 V_1(\eta_i). \qquad (2.49)$$

From the definitions of α and α_1 we have

$$\eta_i = \frac{\alpha}{\alpha_1}\xi_i = \left(\frac{4\sigma}{(n_1+1)\sigma_1}\right)^{1/2}\xi_i \qquad (2.50)$$

and

$$V_1(\eta_i) = \left(\frac{\alpha}{\alpha_1}\right)^3 V(\xi_i) = \left(\frac{4\sigma}{(n+1)\sigma_1}\right)^{3/2} V(\xi_i). \qquad (2.51)$$

In order to solve the equations for η_i, ϕ_i, $V(\eta_i)$ and σ_1, given the values of ξ_i, θ_i, $V(\xi_i)$ and σ in the core we must have a further condition, this maybe afforded by the continuity of ρ, and so from equation (2.41)

$$\theta_i^3 = \phi_i^n + (A_1 - A)\theta_i^4\sigma. \qquad (2.52)$$

From equation (2.52) it follows that in the classical limit ($\sigma \to 0$) equation (2.46) becomes

$$\frac{\sigma_1}{\sigma} = \left(\frac{1}{\theta_i}\right)^{(3-n_1)/n_1} \qquad (2.53)$$

Thus for given values of ξ_i and σ the interfacial values η_i, ϕ_i, $V_1(\eta_i)$ and also σ_1 can be determined. These values provide the necessary (interfacial) boundary conditions to be satisfied in solving equations (2.33) and (2.34).

Numerical results are given in table 1 for small values of σ.

3. Binding energy

The energy of all the constituent particles of the gas dispersed to infinity with zero internal energy is given by

$$E_{0g} = M_{0g}c^2 = \int_0^R 4\pi\rho_g c^2 \, e^{\lambda/2} r^2 \, dr \tag{3.1}$$

where M_{0g}, the rest-mass of the gas can, at least in principle, be calculated by counting the constituent particles and multiplying by the appropriate rest-mass.

We define the binding energy E_b as the difference between the energy of the unbound particles dispersed to infinity with zero internal energy and the total energy of the bound system. Hence

$$E_b = E_{0g} - E$$

or

$$E_b = (M_{0g} - M)c^2 = \int_0^R 4\pi\rho_g c^2 \, e^{\lambda/2} r^2 \, dr - \int_0^R 4\pi\rho c^2 r^2 \, dr. \tag{3.2}$$

In terms of the dimensionless variables ξ, θ, $V(\xi)$, η, ϕ, $V_1(\eta)$, equation (3.2) for the binding energy becomes

$$E_b = 4\pi\rho_{gc}\alpha^3 c^2 \int_0^{\xi_i} \frac{\theta^3 \xi^2 \, d\xi}{(1 - 8\sigma V(\xi)/\xi)^{1/2}} + 4\pi\rho_{gc}\alpha_1^3 c^2 \int_{\eta_i}^{\eta_s} \frac{\phi^{n_1}\eta^2 \, d\eta}{(1 - 2(n_1+1)V_1(\eta)/\eta)^{1/2}} - Mc^2. \tag{3.3}$$

It is not apparent from inspection of equation (3.2) whether the binding energy is a positive or a negative quantity. For, although the gas density ρ_g is smaller than the total density, the factor e^{λ} is in general greater than unity. Consequently the sign of the binding energy can only be ascertained by detailed calculation.

It was pointed out in the introduction that the binding energy plays a fundamental role in determining the stability of a given configuration but before we consider this question we shall analyse in detail the functional dependence of E_b on the central density and the position of the interface.

We begin by noting that, in the particular case when the interface is at the outer surface, so that there is no envelope, equation (3.2) becomes

$$(E_b)_{\xi_s} = 4\pi\rho_{gc}\alpha^3 c^2 \int_0^{\xi_s} \frac{\theta^3 \xi^2 \, d\xi}{(1 - 8\sigma V(\xi)/\xi)^{1/2}} - mc^2 \tag{3.4}$$

where m is the total mass of this model. In the above expression $(E_b)_{\xi_s}$ is just the binding energy of the complete model, the equation of state throughout being given by (2.12). The total mass m is thus given by

$$m = 4\pi\rho_{gc}\alpha^3 V(\xi_s). \tag{3.5}$$

For the difference in the binding energies of the composite model and the complete model we have, using equations (3.3), (3.4), (3.5) and (2.37)

$$E_b-(E_b)_{\xi_s} = 4\pi\rho_{gc}\alpha^3c^2V(\xi_s)-4\pi\rho_{gc}\alpha_1^3c^2V_1(\eta_s)$$
$$-4\pi\rho_{gc}\alpha^3c^2\int_{\xi_i}^{\xi_s}\frac{\theta^3\xi^2\,d\xi}{(1-8\sigma V(\xi)/\xi)^{1/2}}$$
$$+4\pi\rho_{gc}\alpha_1^3c^2\int_{\eta_i}^{\eta_s}\frac{\phi^{n_1}\eta^2\,d\eta}{\{1-2(n_1+1)\sigma_1V_1(\eta)/\eta\}^{1/2}}. \tag{3.6}$$

In this equation we see that, corresponding to each term that refers to the composite configuration, there is a term (in the core variables) that applies to the complete configuration (with a change in sign). Thus the result of any transformation of a composite configuration term can immediately be written down in terms of a similar transformation of the corresponding complete configuration term with the appropriate change of variables.

Thus using equations (2.34) and (2.20) together with (2.47), equation (3.6) becomes

$$E_b-(E_b)_{\xi_s} = 4\pi\rho_{gc}c^2\left\{\alpha_1^3\int_{\eta_i}^{\eta_s}\left(\frac{1}{\{1-2(n_1+1)\sigma_1V_1(\eta)/\eta\}^{1/2}}-1\right)\frac{dV_1}{d\eta}\,d\eta\right.$$
$$-\alpha_1^3\int_{\eta_i}^{\eta_s}\frac{A_1\sigma_1\phi(dV_1/d\eta)\,d\eta}{\{1-2(n_1+1)\sigma_1V_1(\eta)/\eta\}^{1/2}}(1+A_1\sigma_1\phi)^{-1}$$
$$-\alpha^3\int_{\xi_i}^{\xi_s}\left(\frac{1}{(1-8\sigma V(\xi)/\xi)^{1/2}}-1\right)\frac{dV}{d\xi}\,d\xi$$
$$\left.+\alpha^3\int_{\xi_i}^{\xi_s}\frac{A\sigma\theta(dV/d\xi)\,d\xi}{(1-8\sigma V(\xi)/\xi)^{1/2}}(1+A\sigma\theta)^{-1}\right\}. \tag{3.7}$$

On expanding, we find that, in the classical limit

$$E_b^{(1)}-(E_b)_{\xi_s}^{(1)} = 4\pi\rho_{gc}c^2\left(\alpha_1^3(n_1+1)\sigma_1\int_{\eta_i}^{\eta_s}\frac{V_1}{\eta}\frac{dV_1}{d\eta}\,d\eta-4\alpha^3\sigma\int_{\xi_i}^{\xi_s}\frac{V}{\xi}\frac{dV}{d\xi}\,d\xi\right.$$
$$\left.-\alpha_1^3A_1\sigma_1\int_{\eta_i}^{\eta_s}\phi\frac{dV_1}{d\eta}\,d\eta+\alpha^3A\sigma\int_{\xi_i}^{\xi_s}\theta\frac{dV}{d\xi}\,d\xi\right) \tag{3.8}$$

the superscript (1) denoting classical values. This formula gives (in the classical limit) the excess in the binding energy of a composite model over that of the complete (no envelope) model with the same central density, the internal energy being included in the mass density.

Using Appendix 1 together with a similar formula in terms of the core variables and the interfacial boundary conditions in equation (3.8) we obtain

$$E_b^{(1)}-(E_b)_{\xi_s}^{(1)} = 4\pi\rho_{gc}c^2\left\{\alpha_1^3\sigma_1\frac{n_1+1}{3}(3-A_1)\int_{\eta_i}^{\eta_s}\frac{V_1}{\eta}\frac{dV_1}{d\eta}\,d\eta\right.$$
$$-\alpha^3\sigma\frac{4}{3}(3-A)\int_{\xi_i}^{\xi_s}\frac{V}{\xi}\frac{dV}{d\xi}\,d\xi$$
$$\left.+\alpha_1^3\sigma_1\eta_i^3\phi_i^{n_1+1}\left(\frac{A_1}{3}-\frac{A}{3}\right)\right\}. \tag{3.9}$$

If the interface is at the centre of the configuration, equation (3.9) gives

$$E_b^{(1)}-(E_b)_{\xi_s}^{(1)} = 4\pi\rho_{gc}c^2\left\{\sigma_1\alpha_1^3(n_1+1)\left(1-\frac{A_1}{3}\right)\int_0^{\eta_s}\frac{V_1}{\eta}\frac{dV_1}{d\eta}d\eta - 4\sigma\alpha^3\left(1-\frac{A}{3}\right)\int_0^{\xi_s}\frac{V}{\xi}\frac{dV}{d\xi}d\xi\right\} \tag{3.10}$$

and this is just the difference in binding energies of two complete models, one being a configuration for which the equation of state is given by (2.13) and the other being a configuration whose equation of state is given by (2.12). If $\beta \sim 0$, then from (2.12) it follows that $A \sim 3$ and hence the equation of state in the core is approximately identical in form with that of an adiabatic fluid of index 3. Consequently

$$(E_b)_{\xi_s}^{(1)} = 0 \tag{3.11}$$

in accordance with the usual classical result (see Rosseland 1949). Thus for an adiabatic fluid sphere of index n_1 we have $A_1 = 1/(\gamma-1) = n_1$, and (3.10) becomes

$$E_b^{(1)} = 4\pi\rho_{gc}\sigma_1c^2(n_1+1)\left(\frac{3-n_1}{3}\right)\int_0^{\eta_s}\frac{V_1}{\eta}\frac{dV_1}{d\eta}d\eta. \tag{3.12}$$

Hence, in terms of the mass inside coordinate radius r, we have

$$E_b^{(1)} = \frac{3-n_1}{3}\int_0^R \frac{GM_r\,dM_r}{r}$$

which is just the usual expression for the binding energy (in the classical limit) in terms of the gravitational potential energy (see Rosseland 1949, Tooper 1964).

If $\beta \sim 0$ in the core it follows from equation (3.9) that

$$E_b^{(1)} = 4\pi\rho_{gc}\sigma_1c^2\alpha_1^3\left\{\frac{n_1+1}{3}(3-A_1)\int_{\eta_i}^{\eta_s}\frac{V_1}{\eta}\frac{dV_1}{d\eta}d\eta + \eta_i^3\phi_i^{n_1+1}\left(\frac{A_1}{3}-1\right)\right\}. \tag{3.13}$$

In particular, if the envelope corresponds to that of an adiabatic fluid of index n_1 so that $A_1 = n_1$, equation (3.13) gives

$$E_b^{(1)} = 4\pi\rho_{gc}\sigma_1\alpha_1^3c^2\left\{(n_1+1)\left(1-\frac{n_1}{3}\right)\int_{\eta_i}^{\eta_s}\frac{V_1}{\eta}\frac{dV_1}{d\eta}d\eta + \eta_i^3\phi_i^{n_1+1}\left(\frac{n_1}{3}-1\right)\right\}. \tag{3.14}$$

In terms of the dimensionless envelope variables $(\eta, \phi, V_1(\eta))$ equation (A.9) of Appendix 2 becomes

$$\frac{5-n_1}{3}\int_{\eta_i}^{\eta_s}\frac{V_1}{\eta}\frac{dV_1}{d\eta}d\eta = \left(\frac{V_1(\eta)^2}{\eta_s}-\frac{V_1(\eta_i)}{\eta_i}\right) + V_1(\eta_i)\phi_i - \frac{\eta_i^3}{3}\phi_i^{n_1+1}. \tag{3.15}$$

Consequently, equation (3.14) for the binding energy (in the classical limit) of a composite model in which the equation of state in the core is such that (3.11) holds, the equation of state for the envelope being that of an adiabatic fluid of index n_1, becomes

$$E_b^{(1)} = 4\pi\rho_{gc}\sigma_1\alpha_1^3c^2J_{\eta_i}^{(1)} \tag{3.16}$$

where

$$J_{\eta_i}^{(1)} = \frac{(n_1+1)(3-n_1)}{5-n_1}\left\{\left(\frac{V_1(\eta_s)^2}{\eta_s} - \frac{V_1(\eta_i)^2}{\eta_i}\right) + V_1(\eta_i)\phi_i - \frac{2}{n_1+1}\phi_i^{n_1+1}\eta_i^3\right\}. \tag{3.17}$$

We may consider $J_{\eta_i}^{(1)}$ as a 'measure' of the classical binding energy as a function of the position of the interface, given the central rest-density and the central pressure for a given value of n_1. The graph of $J_{\eta_i}^{(1)}$ (for various indices n_1) as a function of the position of the interface is shown in figure 1. For a given n_1 of the envelope we see that the

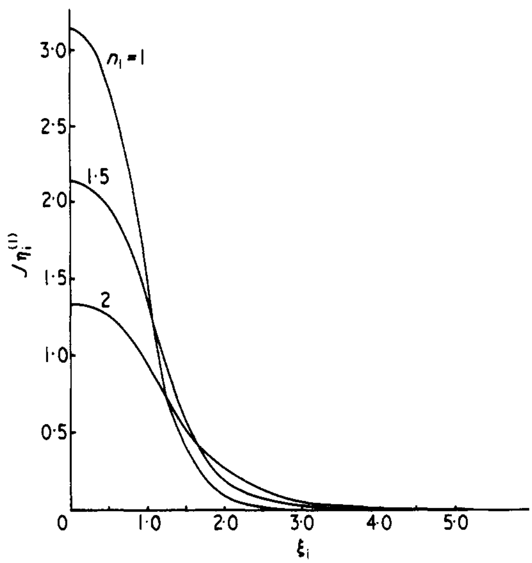

Figure 1. The 'measure' of the classical binding energy $J_{\eta_i}^{(1)}$ defined in equation (3.17) versus the position of the interface (ξ_i) for various values of n_1. The maximum value of each curve is seen to correspond with $\xi_i = 0$ that is when the interface is at the centre and hence the configuration has no core.

binding energy decreases as the position of the interface lies farther from the centre. In newtonian theory the condition for marginal stability of an adiabatic fluid is $\gamma = 4/3$ and the condition for instability is $\gamma < 4/3\,(n_1 > 3)$ (Milne 1930, Rosseland 1949). The condition for instability is equivalent to $E_b \leqslant 0$. In other words in newtonian theory, a negative binding energy is a necessary and sufficient condition for the instability of an adiabatic fluid sphere, and the higher the binding energy the more stable the model. This follows from the fact that the binding energy is the amount of energy required to disperse the constituent particles of the system to infinity against gravity. Thus a system with zero binding energy corresponds to marginal stability, and a tightly bound system has a high binding energy. From figure 1 we see that for a given index n_1 in the envelope, a model for which the interface is nearer the centre is more stable than a similar model (same central pressure and density) with the interface farther from the centre.

When the post-newtonian terms are taken into consideration, instabilities can occur even when the binding energy is positive (Tooper 1964, Fowler 1964). The effect of an envelope on the magnitude of the binding energy will now be considered from the standpoint of general relativity. Equation (3.7) becomes in the post-newtonian approximation

$$
\begin{aligned}
E_\mathrm{b}-(E_\mathrm{b})_{\xi_\mathrm{s}} = 4\pi\rho_{\mathrm{g_c}}c^2\Bigg\{&\alpha_1^3(n_1+1)\sigma_1\int_{\eta_\mathrm{i}}^{\eta_\mathrm{s}}\frac{V_1}{\eta}\frac{\mathrm{d}V_1}{\mathrm{d}\eta}\,\mathrm{d}\eta \\
&+\alpha_1^3\frac{3}{2}(n_1+1)^2\sigma_1^2\int_{\eta_\mathrm{i}}^{\eta_\mathrm{s}}\frac{V_1^2}{\eta^2}\frac{\mathrm{d}V_1}{\mathrm{d}\eta}\,\mathrm{d}\eta-\alpha_1^3A_1\sigma_1\int_{\eta_\mathrm{i}}^{\eta_\mathrm{s}}\frac{\mathrm{d}V_1}{\mathrm{d}\eta}\phi\,\mathrm{d}\eta \\
&-\alpha_1^3A_1\sigma_1\int_{\eta_\mathrm{i}}^{\eta_\mathrm{s}}\frac{\mathrm{d}V_1}{\mathrm{d}\eta}\phi\left((n_1+1)\frac{\sigma_1V_1}{\eta}-A_1\sigma_1\phi\right)\mathrm{d}\eta\Bigg\} \\
&-4\pi\rho_{1_\mathrm{c}}c^2\ \text{(similar terms in core variables)}
\end{aligned}
\qquad (3.18)
$$

from which we obtain

$$
\begin{aligned}
E_\mathrm{b}-(E_\mathrm{b})_{\xi_\mathrm{s}} = {} & 4\pi\rho_{\mathrm{g_c}}\sigma_1\alpha_1^3c^2J_{\eta_\mathrm{i}} \\
&-4\pi\rho_{\mathrm{g_c}}\alpha_1^3\sigma_1^2c^2\Bigg(\frac{n_1+1}{2}\phi_\mathrm{i}^{n_1+1}\eta_\mathrm{i}^2V_1(\eta_\mathrm{i})+(n_1+1)\eta_\mathrm{i}^3\phi_\mathrm{i}^{n_1+2} \\
&+\frac{3}{2}(n_1+1)\int_{\eta_\mathrm{i}}^{\eta_\mathrm{s}}\phi^{2n_1+1}\eta^4\,\mathrm{d}\eta+3(n_1+1)\int_{\eta_\mathrm{i}}^{\eta_\mathrm{s}}\phi^{n_1+2}\eta^2\,\mathrm{d}\eta\Bigg) \\
&+4\pi\rho_{\mathrm{g_c}}\alpha^3\sigma^2c^2\Bigg(2\theta_\mathrm{i}^4\xi_\mathrm{i}^2V(\xi_\mathrm{i})+4\xi_\mathrm{i}^3\theta_\mathrm{i}^5 \\
&+6\int_{\xi_\mathrm{i}}^{\xi_\mathrm{s}}\theta^7\xi^4\,\mathrm{d}\xi+12\int_{\xi_\mathrm{i}}^{\xi_\mathrm{s}}\theta^5\xi^2\,\mathrm{d}\xi\Bigg)
\end{aligned}
\qquad (3.19)
$$

where

$$
J_{\eta_\mathrm{i}} = \left\{\frac{n_1}{3}\eta_\mathrm{i}^3\phi_\mathrm{i}^{n_1+1}+(n_1+1)\left(\frac{n_1}{3}-1\right)\int_{\eta_\mathrm{i}}^{\eta_\mathrm{s}}\phi^{n_1}\eta^3\frac{\mathrm{d}\phi}{\mathrm{d}\eta}\,\mathrm{d}\eta-\frac{\alpha^3}{\alpha_1^3}\frac{\sigma}{\sigma_1}\xi_\mathrm{i}^3\theta_\mathrm{i}^4\right\}. \qquad (3.20)
$$

It is easily verified that in the classical limit (3.19) reduces to equation (3.16).

Using the interfacial boundary conditions, equations (3.19) and (3.20) yield

$$
\begin{aligned}
E_\mathrm{b}-(E_\mathrm{b})_{\xi_\mathrm{s}} = {} & 4\pi\rho_{\mathrm{g_c}}\alpha_1^3\sigma_1c^2J_{\eta_\mathrm{i}}-4\pi\rho_{\mathrm{g_c}}\alpha_1^3\sigma_1c^2(n_1-3)\eta_\mathrm{i}^3\phi_\mathrm{i}^{n_1+3} \\
&-4\pi\rho_{\mathrm{g_c}}\alpha_1^3\sigma_1^2c^2\Bigg(\frac{3}{2}(n_1+1)\int_{\eta_\mathrm{i}}^{\eta_\mathrm{s}}\phi^{2n_1+1}\eta^4\,\mathrm{d}\eta \\
&+3(n_1+1)\int_{\eta_\mathrm{i}}^{\eta_\mathrm{s}}\phi^{n_1+2}\eta^2\,\mathrm{d}\eta\Bigg) \\
&+4\pi\rho_{\mathrm{g_c}}\alpha^3\sigma^2c^2\Bigg(6\int_{\xi_\mathrm{i}}^{\xi_\mathrm{s}}\theta^7\xi^4\,\mathrm{d}\xi+12\int_{\xi_\mathrm{i}}^{\xi_\mathrm{s}}\theta^5\xi^2\,\mathrm{d}\xi\Bigg)
\end{aligned}
\qquad (3.21)
$$

where

$$
J_{\eta_\mathrm{i}} = (n_1+1)\left(\frac{n_1}{3}-1\right)\int_{\eta_\mathrm{i}}^{\eta_\mathrm{s}}\phi^{n_1}\eta^3\frac{\mathrm{d}\phi}{\mathrm{d}\eta}\,\mathrm{d}\eta+\left(\frac{n_1}{3}-1\right)\eta_\mathrm{i}^3\phi_\mathrm{i}^{n_1+1}. \qquad (3.22)
$$

This is the desired expression for the difference in the binding energies of (*a*) a composite model for which the equation of state in the core corresponds to that for a fluid sphere whose equation of state is (2.12) with $\beta \sim 0$, and for which the equation of state in the envelope is that for an adiabatic fluid of index n_1 and (*b*) a complete model for which the equation of state throughout is the same as that of the core in (*a*).

Before giving any quantitative results, we shall give a few simple checks of the above equation. Clearly, when either (i) the interface extends to the surface, or (ii) $n_1 = 3$, it follows that $E_b - (E_b)_{\xi_s} = 0$ as expected. When the interface extends to the centre, that is $\xi_i \to 0$, $\eta_i \to 0$, we have, for the difference between the binding energies of two complete models, one being a sphere with equation of state of the form $p = K_1 \rho_g^{1+1/n_1}$ and the other an adiabatic sphere of index 3

$$\begin{aligned} E_b - (E_b)_{\xi_s} = {} & 4\pi\rho_{g_c}\alpha_1^3\sigma_1 c^2 J_{\eta_s} \\ & - 4\pi\rho_{g_c}\alpha_1^3\sigma_1^2 c^2 \left(\frac{3}{2}(n_1+1) \int_0^{\eta_s} \phi^{2n_1+1}\eta^4 \, d\eta \right. \\ & \left. + 3(n_1+1) \int_0^{\eta_s} \phi^{n_1+2}\eta^2 \, d\eta \right) \\ & + 4\pi\rho_{g_c}\alpha^3\sigma^2 c^2 \left(6 \int_0^{\xi_s} \theta^7 \xi^4 \, d\xi + 12 \int_0^{\xi_s} \theta^5 \xi^2 \, d\xi \right) \end{aligned} \tag{3.23}$$

a massive sphere in which $\beta \sim 0$ corresponds to one with equation of state (2.12) with the classical binding energy zero. Since the third term in equation (3.23) does not depend on n_1 we should expect that the first and second terms correspond to E_b and the third to $(E_b)_{\xi_s}$, namely

$$-(E_b)_{\xi_s} = 4\pi\rho_{g_c}\alpha^3\sigma^2 c^2 \left(6 \int_0^{\xi_s} \theta^7 \xi^4 \, d\xi + 12 \int_0^{\xi_s} \theta^5 \xi^2 \, d\xi \right). \tag{3.24}$$

After simple calculation it can easily be shown that equation (3.24) reduces to

$$-(E_b)_{\xi_s} = \frac{8\pi G}{c^2} \int_0^R pM_r r \, dr + \frac{6\pi G^2}{c^2} \int_0^R \rho M_r^2 \, dr \tag{3.25}$$

which is just the expression for the total energy of a fluid sphere with $\beta \sim 0$ obtained by Fowler (1964). Thus as expected equation (3.24) represents the negative binding energy $-(E_b)_{\xi_s}$ of the complete configuration considered.

Using this result we can readily obtain the binding energy of the composite model under consideration. For from equations (3.21) and (3.24) the binding energy of this model is given by

$$\begin{aligned} E_b = {} & 4\pi\rho_{g_c}\alpha_1^3\sigma_1 c^2 J_{\eta_i} - 4\pi\rho_{g_c}\alpha_1^3\sigma_1^2 c^2 (n_1 - 3)\eta_i^3 \phi_i^{n_1+2} \\ & - 4\pi\rho_{g_c}\alpha_1^3\sigma_1^2 c^2 \left(\frac{3}{2}(n_1+1) \int_{\eta_i}^{\eta_s} \phi^{2n_1+1}\eta^4 \, d\eta \right. \\ & \left. + 3(n_1+1) \int_{\eta_i}^{\eta_s} \phi^{n_1+2}\eta^2 \, d\eta \right) \\ & - 4\pi\rho_{g_c}\alpha^3\sigma^2 c^2 \left(6 \int_0^{\xi_i} \theta^7 \xi^4 \, d\xi + 12 \int_0^{\xi_i} \theta^5 \xi^2 \, d\xi \right) \end{aligned} \tag{3.26}$$

which will be written in the form

$$\frac{E_b}{4\pi\rho_{gc}\alpha_1^3\sigma_1 c^2} = J_{\eta_i} - \sigma_1 \Pi_{n_1}. \tag{3.27}$$

Using expression (2.37) for the total mass M and equation (2.46) we obtain

$$\frac{E_b}{Mc^2} = \frac{\sigma}{V_1(\eta_s)} \frac{\theta_i^4}{\phi_i^{n_1+1}} \left\{ J_{\eta_i} - \sigma \left(\frac{1}{\theta_i}\right)^{(3-n_1)/n_1} \Pi_{n_1} \right\}. \tag{3.28}$$

4. Numerical results

In solving the equations of equilibrium for various positions of the interface and various values of n_1 we assume that β is extremely small in the core. From consideration of the graphs given by Tooper (1965) for the binding energy of complete models the maximum binding energy of a composite configuration as a function of σ may be expected to occur for small values of σ and so the equations of hydrostatic equilibrium (2.33) and (2.34) are solved for various positions of the interface ξ_i subject to the boundary conditions and assuming that $\sigma \ll 1$.

In figures 2 and 3 the binding energy per unit mass is displayed as a function of the parameter σ for various positions of the interface (ξ_i), n_1 being 1 in figure 2 and 2 in figure 3. We see that for a given n_1 in the envelope the binding energy decreases with increasing ξ_i for the range of values considered. In other words for a given value of n_1 and σ the nearer the interface is to the centre the larger the binding energy.

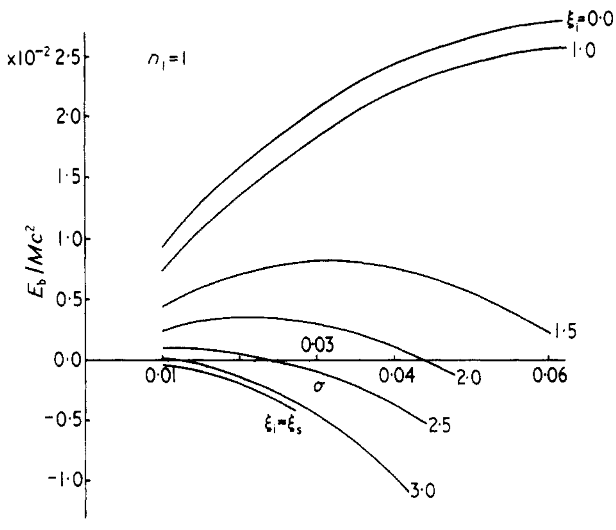

Figure 2. The dimensionless binding energy given by equation (3.28) versus the parameter σ for $n_1 = 1$ in the envelope for various positions of the interface ξ_i.

For complete models, using Chandrasekhar's variational principle (1964), Tooper (1969, 1965) has shown that instability sets in at the first peak of the binding energy as a function of σ. Assuming that the same is true for composite models it would mean that in figures 2 and 3 instability would occur at the value of σ for which the binding energy is a maximum for a given model and the model is unstable for larger values of σ, even though the binding energy is positive.

The total energy of a fluid sphere exclusive of the rest-mass energy when infinitely dispersed from its equilibrium state is equal in magnitude but opposite in sign to the binding energy and this allows us to give a simple explanation of the onset of instability at the maximum of the binding energy regarded as a function of σ. Suppose σ_m is the value of σ at which the binding energy is a maximum or at which the internal energy required for hydrostatic equilibrium is a minimum. Then, if we consider the adiabatic expansion of a model for which $\sigma > \sigma_m$, the binding energy would be increased, in other words, the equilibrium energy required after expansion would be less than that required before the expansion and so further expansion would ensue. On the other hand for a model for which $\sigma < \sigma_m$, the opposite is true: after expansion more internal energy would be required to maintain equilibrium but since this is not forthcoming (it being

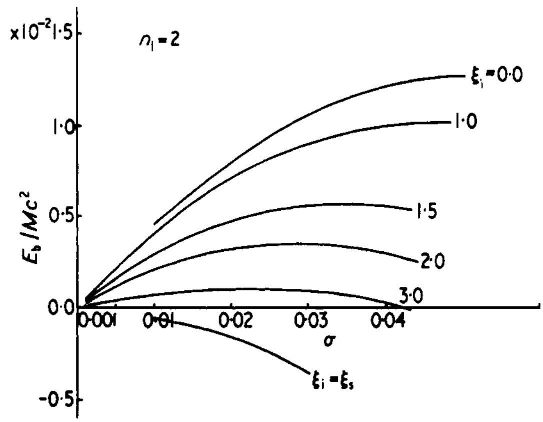

Figure 3. The dimensionless binding energy given by equation (3.28) versus the parameter σ for $n_1 = 2$ in the envelope for various positions of the interface ξ_i.

assumed that there is no energy generation in the core) the expansion stops. Consider next adiabatic contraction. In a configuration for which $\sigma > \sigma_m$, the binding energy would be reduced and hence the energy required for hydrostatic equilibrium would be increased; since this energy is not made available in the adiabatic contraction further collapse must ensue. Again, for a configuration for which $\sigma < \sigma_m$ the opposite would be the case. Following contraction, less internal energy would be required to maintain equilibrium, and since this excess energy cannot be emitted, contraction stops. Thus we see that σ_m, the value of σ corresponding to maximum binding energy, may be regarded as the critical value of σ at which instability sets in.

From figures 2 and 3 we can also see how the position of the interface affects stability. For a given n_1 in the envelope, as ξ_i increases (ie the model consists of more and more core) the maximum of the binding energy as a function of σ moves to the left of the diagram, that is it occurs for smaller values of σ. Moreover for large values of ξ_i (ie ξ_i close to ξ_s), the binding energy is always negative. For $n_1 = 3$ (or equivalently $\xi_i = \xi_s$) the classical binding energy is zero and the post-newtonian terms are negative. Thus in this case the binding energy is always negative and these objects are unstable over the full range of values of σ. But even in the case of small ξ_i the models can become unstable, for sufficiently large values of σ, even when the binding energy is positive.

The application of an envelope to a core (for which $n_1 = 3$) has the effect of increasing the binding energy and produces a peak in the graph representing it as a function of σ. The smaller the interfacial radius the higher is this peak and the larger the value of σ_m at which it occurs. For a given ξ_i we find that the smaller the value of n_1 the larger the value of σ_m at which the peak in the binding energy occurs.

From the above considerations we can draw the following general conclusions. Given a core consisting of matter and radiation in which β, the ratio of the gas pressure to the total pressure, is an extremely small constant and an envelope fitted onto this core subject to the usual boundary conditions, the envelope being an adiabatic spherical shell of index $n_1 < 3$, we conclude that the envelope has a significant effect on the stability of the whole system in the sense that, the smaller the interfacial radius, the greater is the range of central density compatible with stability.

Appendix 1

Let

$$I_{n_1}(\eta_s) = (n_1+1)\int_{\eta_i}^{\eta_s} \frac{V_1}{\eta}\frac{\mathrm{d}V_1}{\mathrm{d}\eta}\,\mathrm{d}\eta - A_1\int_{\eta_i}^{\eta_s} \phi\frac{\mathrm{d}V_1}{\mathrm{d}\eta}\,\mathrm{d}\eta. \tag{A.1}$$

Using equation (2.34) in the classical limit, that is, $\mathrm{d}V_1/\mathrm{d}\eta = \phi^{n_1}\eta^2$, we have

$$A_1\int_{\eta_i}^{\eta_s} \phi\frac{\mathrm{d}V_1}{\mathrm{d}\eta}\,\mathrm{d}\eta = A_1\int_{\eta_i}^{\eta_s} \phi^{n_1+1}\eta^2\,\mathrm{d}\eta$$

and on integrating by parts we find that

$$A_1\int_{\eta_i}^{\eta_s} \phi\frac{\mathrm{d}V_1}{\mathrm{d}\eta}\,\mathrm{d}\eta = \frac{A_1}{3}[\eta^3\phi^{n_1+1}]_{\eta_i}^{\eta_s} - A_1\frac{n_1+1}{3}\int_{\eta_i}^{\eta_s} \phi^{n_1}\eta^3\frac{\mathrm{d}\phi}{\mathrm{d}\eta}\,\mathrm{d}\eta.$$

Using the generalization of the Lane–Emden equation (2.33) in the classical limit we obtain

$$A_1\int_{\eta_i}^{\eta_s} \phi\frac{\mathrm{d}V_1}{\mathrm{d}\eta}\,\mathrm{d}\eta = -\frac{A_1}{3}\eta_i^3\phi_i^{n_1+1} + A_1\frac{n+1}{3}\int_{\eta_i}^{\eta_s} \frac{V_1}{\eta}\frac{\mathrm{d}V_1}{\mathrm{d}\eta}\,\mathrm{d}\eta$$

where we have used the condition that $\phi(\eta_s) = 0$ at the surface. Hence (A.1) becomes

$$I_{n_1}(\eta_s) = (n_1+1)\left(1-\frac{A_1}{3}\right)\int_{\eta_i}^{\eta_s} \frac{V_1}{\eta}\frac{\mathrm{d}V_1}{\mathrm{d}\eta}\,\mathrm{d}\eta + \frac{A_1}{3}\eta_i^3\phi_i^{n_1+1}. \tag{A.2}$$

Appendix 2. Derivation of the gravitational potential energy in the envelope of a composite model

We calculate the gravitational potential energy (in the classical limit) Ω_i of the outer part of the model (the envelope) between the interface $r = r_i$ and the surface $r = R$. Thus

$$-\Omega_i = G\int_{r_i}^{R} \frac{M_r\,\mathrm{d}M_r}{r} = \frac{1}{2}G\left(\frac{M^2}{R} - \frac{M_i^2}{r_i}\right) + \frac{1}{2}G\int_{r_i}^{R} \frac{M_r^2}{r^2}\,\mathrm{d}r. \tag{A.3}$$

Defining S_r by

$$\frac{\mathrm{d}S_r}{\mathrm{d}r} = \frac{GM_r}{r^2}$$

we obtain

$$-\Omega_i = \frac{1}{2}G\left(\frac{M^2}{R}-\frac{M_i^2}{r_i}\right)+\frac{1}{2}\int_{r_i}^{R}\frac{dS_r}{dr}M_r\,dr$$

and hence

$$-\Omega_i = \frac{1}{2}G\left(\frac{M^2}{R}-\frac{M_i^2}{r_i}\right)-\frac{1}{2}\frac{GM^2}{R}-\frac{1}{2}S_iM_i-\frac{1}{2}\int_{r_i}^{R}S_r\,dM_r$$

where we have used the formula $S_R = -GM/R$ where subscript i, as before, denotes interfacial values. Consequently

$$-\Omega_i = -\frac{1}{2}G\frac{M_i^2}{r_i}-\frac{1}{2}S_iM_i-\frac{1}{2}\int_{r_i}^{R}S\,dM_r. \tag{A.4}$$

In the classical limit

$$-\frac{dS_r}{dr} = -\frac{GM_r}{r^2} = \frac{1}{\rho}\frac{dp}{dr} = (n_1+1)\frac{d}{dr}\left(\frac{p}{\rho}\right)$$

and so on integrating we have

$$-S_r+S_R = (n_1+1)\frac{p}{\rho}.$$

Hence

$$-S_r = (n_1+1)\frac{p}{\rho}+\frac{GM}{R} \tag{A.5}$$

and thus

$$-S_i = (n_1+1)\frac{p_i}{\rho_i}+\frac{GM}{R}. \tag{A.6}$$

Consequently, on using equations (A.4), (A.5) and (A.6), we obtain

$$-\Omega_i = -\frac{1}{2}\frac{GM_i^2}{r_i}+\frac{1}{2}(n_1+1)\frac{p_i}{\rho_i}M_i+\frac{1}{2}G\frac{M}{R}M_i+\frac{1}{2}(n_1+1)\int_{r_i}^{R}\frac{p}{\rho}dM_r$$
$$+\frac{1}{2}\frac{GM}{R}\int_{r_i}^{R}dM_r$$

and so

$$-\Omega_i = -\frac{1}{2}\frac{GM_i^2}{r_i}+\frac{1}{2}\frac{GM^2}{R}+\frac{1}{2}(n_1+1)\frac{p_i}{\rho_i}M_i+\frac{1}{2}(n_1+1)\int_{r_i}^{R}p\,dV \tag{A.7}$$

where $dV = 4\pi r^2\,dr$.

Also

$$-\Omega_i = G\int_{r_i}^{R}\frac{M_r\,dM_r}{r} = -4\pi\int_{r_i}^{R}\frac{dp}{dr}r^3\,dr = -4\pi[pr^3]_{r_i}^{R}+12\pi\int_{r_i}^{R}pr^2\,dr.$$

Hence

$$-\Omega_i = 3p_i V_i + 3\int_{r_i}^{R} p\,dV. \tag{A.8}$$

Using (A.8) in (A.7) we have

$$-\left(\frac{5-n_1}{3}\right)\Omega_i = G\left(\frac{M^2}{R}-\frac{M_i^2}{r_i}\right)+(n_1+1)\frac{p_i}{\rho_i}M_i-(n_1+1)p_i V_i. \tag{A.9}$$

This is the desired formula for the gravitational potential energy of the envelope. It may be noted that, in the particular case when the interface is at the centre of the model

$$\Omega = \frac{3}{n_1-5}\frac{GM^2}{R} \tag{A.10}$$

which is the usual expression for the gravitational potential energy of an adiabatic fluid sphere (or a polytrope) of index n_1 (Fowler 1964 and Tooper 1965b).

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